

Help Notes: Integration

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Antiderivative

Definition 0.1. A function F is called an **antiderivative of f** on an interval I if $F'(x) = f(x)$ for all x in I .

Theorem 0.2. If F is an antiderivative of f on an interval I , then any antiderivative of f on an interval I is of the form

$$F(x) + C$$

when C is an arbitrary constant.

Finding an antiderivative of f is finding a function F whose derivative is f , there are infinitely many antiderivative for a given function they are equal up to a constant. Giving all the antiderivative for a function is giving a particular antiderivative of a function and add a constant that you let vary over all the real number.

To find an antiderivative, you can just use your derivative table from the right to the left here some example. In the following F denote an antiderivative for f and G an antiderivative for g

Function	Particular antiderivative
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$1/x$	$\ln x $
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\csc^2(x)$	$-\cot(x)$
$\csc(x)\cot(x)$	$-\csc(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$
$1/x$	$\ln x $
a^x	$\frac{a^x}{\ln(a)}$
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$g'(x)f(g(x))$	$F(g(x))$

You do not have to learn this table, you can just try to force the derivative to appear and use the table of the derivative, you can always differentiate your answer after word to make sure that your answer is correct

For instance,

$$f(x) = x^4 + x^3 + 5x^2 + 6$$

You can write

$$f(x) = \frac{1}{5} \cdot 5x^4 + \frac{1}{4}4x^3 + \frac{5}{3}3x^2 + 6 \cdot 1$$

Then, if F is any antiderivative for f

$$F(x) = \frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{5}{3}x^3 + 6 \cdot x + c$$

where c varies through the real numbers

Another example :

$$f(x) = 2x \sqrt{5x^2 + 3}$$

You can identify a composite thus it seem that this could be coming from a chain rule maybe, but if this is the case you should be able to write

$$f(x) = ku'(x)v'(u(x))$$

You need now to identify what functions play the role of u' , v' , u and the constant k ,
It seems that

$$u(x) = 5x^2 + 3 \text{ so that } u'(x) = 10x$$

Also, if we take

$$v'(x) = \sqrt{x} \text{ so that we could have } v(x) = 2/3x^{3/2}$$

Look we get

$$u'(x)v'(u(x)) = 10x \sqrt{5x^2 + 3}$$

So that

$$f(x) = \frac{1}{5}u'(x)v'(u(x))$$

So if you want now any antiderivative of f F it will be

$$F(x) = \frac{1}{5}v(u(x)) = \frac{2}{15}(5x^2 + 3)^{3/2}$$

So basically you force the derivative to appear so you can use the derivative table from right to left.

Integral

Definition 0.3. If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0(=a), x_1, x_2, \dots, x_n(=b)$ be the endpoints of these subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then, the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

provided that this limit exists. if it does exist, we say that f is **integrable** on $[a, b]$.

Note that the symbol \int was introduced by Leibniz and is called an **integral sign**. $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. The dx simply indicates that the dependent variable is x . $\int_a^b f(x)dx$ is all one symbol. The procedure of calculating an integral is called **integration**. We can write $\int f(x)dx$ for the antiderivative of f when x is chosen to be the variable.

Note that

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

The name of the variable do not change the value of the integral.

If I ask you something in term of an area please be careful to check the sign of f because of the following

- If $f(x) \geq 0$, the integral $\int_a^b f(x)dx$ is the area delimited by the graph of f , the x -axis and the line $x = a$ and $x = b$.
- When $f(x) \leq 0$ the integral $\int_a^b f(x)dx$ is minus the area delimited by the graph of f , the x -axis and the line $x = a$ and $x = b$.
- In general, $\int_a^b f(x)dx$ is the sum of the areas above the x -axis minus the sum of the areas below the x -axis (for the domains delimited by the graph of f , the x -axis and the line of equation $x = a$ and $x = b$).

Theorem 0.4. If f is continuous on $[a, b]$, or f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is the definite integral $\int_a^b f(x)dx$ exists.

Properties of integrals : TO BE REMEMBERED WELL!!!!

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b cdx = c(b-a)$, where c is any constant.
4. $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ (We will refer to this property as linearity property of the integral)
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant. (We will also refer to this property as linearity property of the integral)
6. $\int_a^b (f(x) - g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx$. (We will also refer to this property as linearity property of the integral)

7. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$ (We will refer to this property as additive property of the integral)
8. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$ (We will refer to this property as comparison property of the integral)
9. If $f(x) \geq g(x)$, for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$. (We will refer to this property as comparison property of the integral)
10. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

(We will refer to this property as comparison property of the integral)

Theorem 0.5. *If f is continuous on the interval $[a, b]$, then*

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where F is any antiderivative of f , that is, $F' = f$.

If you can use this theorem then it only about finding an antiderivative that we have discussed in the previous section and evaluating it at b and a .

Definition 0.6. *The notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called an **indefinite integral**. Thus*

$$\int f(x)dx = F(x) \text{ means } F'(x) = f(x)$$

We have also the Fundamental theorem of calculus that says that

Theorem 0.7. *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is an antiderivative of f , that is $g'(x) = f(x)$ for $a < x < b$.

Basically if I ask you : Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}dt$. You should be super happy. We have $g'(x) = 1 + x^2$ from the previous theorem. Be careful of course it is x not t !!!

If I ask you : Find

$$\frac{d}{dx} \int_1^{x^4} \sec(t)dt$$

Well it is a bit harder but not much more, we have that

$$\int_1^{x^4} \sec(t)dt = g(x^4)$$

where $g(x) = \int_1^x \sec(t)dt$, then $g'(x) = \sec(x)$ from the previous theorem. and for $\int_1^{x^4} \sec(t)dt$ it is just a bit of chain rule

$$\frac{d}{dx} \int_1^{x^4} \sec(t)dt = 4x^3 g'(x^4) = 4x^3 \sec(x^4)$$

Not so bad....

Theorem 0.8 (Substitution rule). *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then*

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Theorem 0.9 (Substitution rule). *If g' is continuous on $[a, b]$ and f is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Let's do one example even if it is very similar to what we have done in the antiderivative section

$$\int 2x \sqrt{1+x^2} dx$$

is by definition any antiderivative of $2x \sqrt{1+x^2}$, so it is just like the first section you see that you have some kind of composite so maybe it is a substitution rule that is going on We want to see $2x \sqrt{1+x^2} = kg'(x)f'(g(x))$ where we need to identify the function f' , g and the constant k .

You see with a bit of practice that certainly a good choice is to take $g(x) = 1+x^2$ then $g'(x) = 2x$ and $f'(x) = \sqrt{x}$ so that f could be $f(x) = 2/3 x^{3/2}$ But then

$$g'(x)f'(g(x)) = 2x \sqrt{1+x^2}$$

And finally, using the substitution rule we get

$$\int 2x \sqrt{1+x^2} dx = 2/3(1+x^2)^{3/2} + c$$

where c could be any real number.

Lets do another definite one just to make sure you with me

$$\int_1^2 \frac{dx}{(3-5x)^2}$$

We identify a composite again maybe a chain rule could be a good idea to compute the integral that we want. For this we want to write

$$\frac{1}{(3-5x)^2} = ku'(x)v'(u(x))$$

where u, v' are function to be identify and k is a constant to be identified.

With a bit of practice you will see that it is not hard to see that a good choice for u is

$$u(x) = 3 - 5x \text{ then we have } u'(x) = -5$$

and

$$v'(x) = \frac{1}{x^2} \text{ so that we could take } v(x) = -\frac{1}{x}$$

So that

$$u'(x)v'(u(x)) = -5\frac{1}{(3-5x)^2}$$

and thus

$$\frac{1}{(3-5x)^2} = \frac{-1}{5}u'(x)v'(u(x))$$

Now we have by the substitution rule :

$$\begin{aligned} \int_1^2 \frac{dx}{(3-5x)^2} &= \int_1^2 \frac{-1}{5}u'(x)v'(u(x))dx = -1/5[v(u(x))]_1^2 \\ &= -1/5\left(\frac{-1}{3-5 \cdot 2} - \frac{-1}{3-5 \cdot 1}\right) \\ &= -1/5\left(\frac{1}{7} - \frac{1}{2}\right) \\ &= -1/5 \frac{-5}{14} = 1/14 \end{aligned}$$

You have plenty of other example in the tutorial and book if you want to see more of this.